CESÀRO SUMS AND ALGEBRA HOMOMORPHISMS OF BOUNDED OPERATORS

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Abstract. Let X be a complex Banach space. The connection between algebra homomorphisms defined on subalgebras of the Banach algebra $\ell^1(\mathbb{N}_0)$ and the algebraic structure of Cesàro sums of a linear operator $T \in \mathcal{B}(X)$ is established. In particular, we show that every (C,α)-bounded operator T induces - and is in fact characterized - by such an algebra homomorphism. Our method is based on some sequence kernels, Weyl fractional difference calculus and convolution Banach algebras that are introduced and deeply examined. To illustrate our results, improvements to bounds for Abel means, new insights on the (C,α) boundedness of the resolvent operator for temperated α-times integrated semigroups, and examples of bounded homomorphisms are given in the last section.

1. Introduction

Let X be a complex Banach space. Let T be an operator in the Banach algebra $\mathcal{B}(X)$ and denote by $\mathcal{T}$ the discrete semigroup given by $\mathcal{T}(n) := T^n$ for $n \in \mathbb{N}_0$. The Cesàro sum of order $\alpha > 0$ of T, $\{\Delta^{-\alpha}\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$, is defined by

$$\Delta^{-\alpha}\mathcal{T}(n)x = \sum_{j=0}^{n} k^{\alpha}(n-j)\mathcal{T}(j)x, \qquad x \in X; \quad n \in \mathbb{N}_0,$$

where

$$k^{\alpha}(n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}, \qquad n \in \mathbb{N}_0,$$

is the Cesàro kernel. It is well known that Cesàro sums is an important concept that appears in several contexts and ways in the literature. For instance, in Zygmund's book, it appeared in connection with summability of Fourier series [29, Chapter III, Section 3.11] and in [7] in relation with weighted norm inequalities for Jacobi polynomials and series. See also [20] and [24]. The starting point for our investigation is this definition of fractional sum of the discrete semigroup $\mathcal{T}$. Certain fractional sums have been used in recent years to develop a theory of fractional differences with interesting applications to boundary value problems and concrete models coming from biological issues, see for example [5] and [19]. Note that this definition

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coincides or is connected with other fractional sums of the discrete semigroup $\mathcal{T}$ on the set $\mathbb{N}_0$, see [4, Section 1] or [6, Theorem 2.5].

Consider $\phi : \mathbb{N}_0 \to \mathbb{R}^+$ a positive weight sequence and the Banach algebra ℓ^1_ϕ (endowed with their natural convolution product). Suppose $\frac{1}{\phi(\cdot)}\mathcal{T} \in \ell^\infty(\mathcal{B}(X))$. It is well known and easy to show that the semigroup $\mathcal{T}$ induces an algebra homomorphism $\theta : \ell^1_\phi \to \mathcal{B}(X)$ defined by

$$\theta(f)x := \sum_{n=0}^{\infty} f(n)\mathcal{T}(n)x, \qquad f \in \ell^1_\phi, \quad x \in X.$$

Note that in the case that T is a power bounded operator, i.e., $\mathcal{T} \in \ell^\infty(\mathcal{B}(X))$, then $\theta : \ell^1 \to \mathcal{B}(X)$. Moreover, this homomorphism is a natural extension of the Z-transform, see for example [12] and references therein.

In general, algebra homomorphisms are useful tools to treat different interesting aspects of operator theory: Algebra relations, sharp norm estimations, subordination operators, or ergodic behaviour (as Katznelson-Tzafriri theorems, see [22]).

As mentioned before, it is remarkable that Cesàro sums have appeared in the literature since some time ago but until now there was not noted their relationship with the theory of fractional sums and their algebraic structure. The first main purpose of this paper is to show how this connection provide new insight on properties and characterizations of Cesàro sums, notably concerning their interplay with algebra homomorphisms.

Cesàro sums are also a basic tool to define (C, α)-bounded operators, a natural extension of power-bounded operators. We recall that a bounded operator $T \in \mathcal{B}(X)$ is (C, α)-bounded $(\alpha > 0)$ if

$$\sup_n \| \frac{1}{k^{\alpha+1}(n)} \Delta^{-\alpha}\mathcal{T}(n)\| < \infty.$$

See [10, 26] for examples and properties of (C, α)-bounded operators. Note that if T is power bounded, then T is a (C, α)-bounded operator for every $\alpha > 0$. However, there are operators that does not satisfy the power-boundedness condition, but $\sup_{n\geq 1} \frac{1}{n}\|\Delta^{-1}\mathcal{T}(n)\| < \infty$, as the well-known Assani example shows

$$T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix},$$

see [13, Section 4.7]; recently other examples are appeared in [10, 11, 26, 27, 28].

The following natural question then arises: (Q) Can T induce an algebra homomorphism from a proper subalgebra $\mathcal{A} \subset \ell^1$ to $\mathcal{B}(X)$ such that Cesàro sums are kernels of this homomorphism?.

The second purpose of this paper is to show that, surprisingly, the answer to (Q) is positive for every bounded operator such that their Cesàro sums are properly bounded (which includes (C, α)-bounded operators). More precisely, we construct appropriate subalgebra $\tau^\alpha(k^{\alpha+1}) \subset \ell^1$ and then we prove that the following assertions are equivalent:

(i) T is (C, α)-bounded operator.
(ii) There exists a bounded algebra homomorphism $\theta : \tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.

In the limit case, the following assertions are equivalent:

(a) T is power bounded.
(b) There exists a bounded algebra homomorphism $\theta : \ell^1 \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.

(c) For any $0 < \alpha < 1$, there exist bounded algebra homomorphisms $\theta_\alpha : \tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ such that $\theta_\alpha(e_1) = T$ and $\sup_{0<\alpha<1} \|\theta_\alpha\| < \infty$.

This paper is organized as follows: In order to construct a suitable Banach algebra and the corresponding homomorphism, we introduce in Section 2 the notion of α-th fractional Weyl sum as follows:

$$W^{-\alpha} f(n) = \sum_{j=n}^{\infty} k^\alpha(j-n) f(j), \qquad n \in \mathbb{N}_0.$$

see Definition 2.2 below. We state their main algebraic properties in Proposition 2.3. Then, we introduce Banach algebras $\tau^\alpha(\phi)$ as the completion of the space of sequences $c_{0,0}$ under the norm $q_\phi(f) := \sum_{n=0}^{\infty} \phi(n)|W^\alpha f(n)|$ (Theorem 2.10). The weighted sequences ϕ need to verify some summability conditions (Definition 2.7) to prove that the space $\tau^\alpha(\phi)$ is a Banach algebra. It is remarkable that such Banach algebras extends those defined for $\alpha \in \mathbb{N}_0$ and $\phi = k^{\alpha+1}$ in [17, Section 4]. There they are considered to study subalgebras of analytic functions on the unit disc contained in the Koremblyum and (analytic) Wiener algebra.

Section 3 contains an interesting characterization for the Cesàro sum of powers of a given (C, α)-bounded operator $T \in \mathcal{B}(X)$ solely in terms of certain functional equation (Theorem 3.3). The obtained characterization corresponds to an extension of the well known functional equation for the corresponding discrete semigroup $\mathcal{T}$, namely

$$T^n T^m = T^{n+m}, \quad n, m \in \mathbb{N}_0.$$

Theorem 3.5 gives a complete answer to question (Q) by defining a bounded algebra homomorphism $\theta : \tau^\alpha(\phi) \to \mathcal{B}(X)$ given explicitly by

$$\theta(f)x := \sum_{n=0}^{\infty} W^\alpha f(n) \Delta^{-\alpha} \mathcal{T}(n)x, \qquad f \in \tau^\alpha(\phi), \quad x \in X.$$

This homomorphism enjoys remarkable properties. The existence of bounded homomorphisms in these new Banach algebras completely characterizes the growth of Cesàro sums in Corollary 3.6; in particular bounded homomorphisms from algebras $\tau^\alpha(k^{\alpha+1})$ characterizes (C, α)-boundedness (Corollary 3.7). Such connection seems to be new in the current literature as well as the functional equation found in the beginning of this section.

The Z-transform technique may be traced back to De Moivre around the year 1730. In fact, De Moivre introduced the more general concept of "generating functions" to probability theory. It is interesting compare the Z-transform (discrete case) versus Laplace transform (continuous case), see for example [12, Section 6.7]. In Section 4, we use the Widder space $C_W^\infty((\omega, \infty), X; \mathrm{m})$ where m is Borel measure on $\mathbb{R}_+$, introduced in [8], to give a new characterization of summable vector-valued sequences in in terms of Z-transform in Theorem 4.1. We complete the approach given in Section 3 involving the Z-transform and resolvent operators in Theorem 4.4.

Finally, in Section 5 we present several applications, counterexamples and final comments on this paper. A straightforward application is to obtain the Abel means by subordination to the Cesàro sums, as Theorem 5.1 shows. This point of view allows to improve some previous results given in [25]. Some results presented in this paper are inspired in similar ones obtained for α-times integrated semigroups, see [15]. In Section 5.2, we show a natural connection between both operator theories. In Section 5.3, we present some counterexamples of algebra homomorphisms

defined from some Banach algebras which cannot be extended to some larger algebras. A future research line, the extension of celebrated Katznelson-Tzafriri to (C,α)-bounded operators, is commented in Section 5.4.

Notation. We denote by $\{e_n\}_{n\in\mathbb{N}_0}$ the set of canonical sequences given by $e_n(j)=\delta_{n,j}$ where $\delta_{n,j}$ is the known Kronecker delta, i.e., $\delta_{n,j}=1$ is $n=j$ and 0 in other case. Let X be a Banach space and $\ell^p(X)$ the set of vector-valued sequences $f:\mathbb{N}_0\to X$ such that $\displaystyle\sum_{n=0}^{\infty}\|f(n)\|^p<\infty$, for $1\le p<\infty$; and $c_{0,0}(X)$ the set of vector-valued sequences with finite support. When $X=\mathbb{C}$ we write ℓ^p and $c_{0,0}$ respectively. It is well known that ℓ^1 is a Banach algebra with the usual (commutative and associative) convolution product

$$(f*g)(n)=\sum_{j=0}^{n}f(n-j)g(j),\qquad n\in\mathbb{N}_0.$$

Consider $\phi:\mathbb{N}_0\to\mathbb{R}^+$ a positive sequence, and ℓ^1_ϕ is the Banach spaced formed by complex sequences $f:\mathbb{N}_0\to\mathbb{C}$ such that $\sum_{n\in\mathbb{N}_0}\phi(n)|f(n)|<\infty$. We write $f^{*n}=f*f^{*(n-1)}$ for $n\ge 2$, $f^{*1}=f$ and $f^{*0}=e_0$; in particular $e_n=e_1^{*n}$ for $n\in\mathbb{N}_0$.

Throughout the paper, we use the variable constant convention, in which C denotes a constant which may not be the same from line to line. The constant is frequently written with subindexes to emphasize that it depends on some parameters.

2. Weyl differences and convolution Banach algebras

In this section, we define certain spaces of sequences that corresponds to an extension in two different directions of those considered in the recent paper [17, Definition 4.2]. We consider a positive order of regularity in Weyl differences (Definition 2.2) and different order of growth of Weyl differences (Definition 2.7). These spaces correspond to Banach subalgebras of the space ℓ^1 and are important to obtain a further characterization via homomorphisms for Cesàro sums in the next section.

We consider the usual difference operator $\Delta f(n)=f(n+1)-f(n)$, for $n\in\mathbb{N}_0$, its powers $\Delta^{k+1}=\Delta^k\Delta=\Delta\Delta^k$, for $k\in\mathbb{N}$, and we write by $\Delta^0 f=f$ and $\Delta^1=\Delta$. It is easy to see that

$$\Delta^k f(n)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(n+j),\qquad n\in\mathbb{N}_0,$$

see for example [12, (2.1.1)] and then $\Delta^m:c_{0,0}\to c_{0,0}$ for $m\in\mathbb{N}_0$. In addition, for $\alpha>0$, we consider the well-known scalar sequence $(k^\alpha(n))_{n=0}^{\infty}$ defined by

$$k^\alpha(n):=\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}=\binom{n+\alpha-1}{\alpha-1},\qquad n\in\mathbb{N}_0.$$

In the classical Zygmund's monographic, the numbers $k^\alpha(n)$ are called as Cesàro numbers of order α ([29, Vol. I, p.77]) and written by $k^\alpha(n)=A_n^{\alpha-1}$. However the notation as function k^α will facilitate the understanding of this paper. Kernels k^α may equivalently be defined by means of the generating function:

$$\sum_{n=0}^{\infty}k^\alpha(n)z^n=\frac{1}{(1-z)^\alpha},\quad |z|<1,\quad \alpha>0,\tag{2.1}$$

and satisfies the semigroup property, that is, $k^\alpha * k^\beta = k^{\alpha+\beta}$ for $\alpha, \beta > 0$. Furthermore, the following equality holds: for $\alpha > 0$,

$$k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1 + O(\frac{1}{n})), \qquad n \in \mathbb{N}, \tag{2.2}$$

([29, Vol. I, p.77 (1.18)]) and k^α is increasing (as a function of n) for $\alpha > 1$, decreasing for $1 > \alpha > 0$ and $k^1(n) = 1$ for $n \in \mathbb{N}$ ([29, Theorem III.1.17]). It is straightforward to check that $k^\alpha(n) \le k^\beta(n)$ for $\beta \ge \alpha > 0$ and $n \in \mathbb{N}_0$. The Gautschi inequality states that

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}, \qquad x \ge 1, \quad 0 < s < 1, \tag{2.3}$$

([18]), which implies that

$$\frac{(n+1)^{\alpha-1}}{\Gamma(\alpha)} < k^\alpha(n) < \frac{n^{\alpha-1}}{\Gamma(\alpha)}, \qquad n \in \mathbb{N}, \quad 0 < \alpha < 1.$$

Note that when $\alpha = 0$ we have

$$k^0(n) := \lim_{\alpha \to 0^+} k^\alpha(n) = e_0(n), \qquad n \in \mathbb{N}_0.$$

Lemma 2.1. *For $\alpha > 0$, there exists $C_\alpha > 0$ such that*

$$k^\alpha(2n) \le C_\alpha k^\alpha(n), \qquad n \in \mathbb{N}_0.$$

In particular for $0 < \alpha < 1$, the following equality holds

$$k^{\alpha+1}(2n) < 2^\alpha k^{\alpha+1}(n)\left(1 + \frac{1-\alpha}{2(1+\alpha)}\right)^\alpha, \qquad n \in \mathbb{N}_0.$$

Proof. The proof of the first inequality is straightforward by the inequality (2.2). To show the second inequality, we use the know doubling equality for Gamma function

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \qquad \Re z > 0,$$

to obtain that

$$k^{\alpha+1}(2n) \;=\; \frac{\Gamma(\alpha+1+2n)}{\Gamma(\alpha+1)\Gamma(2n+1)} = 2^\alpha k^{\alpha+1}(n)\frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2}+n)\Gamma(\frac{\alpha}{2}+1+n)}{\Gamma(\alpha+1+n)\Gamma(\frac{1}{2}+n)}, \qquad n \ge 1.$$

We apply the Gautschi inequality (2.3) to get that

$$\begin{aligned} \frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2}+n)}{\Gamma(\frac{1}{2}+n)} &< (\frac{\alpha}{2}+\frac{1}{2}+n)^{\frac{\alpha}{2}}, \\ \frac{\Gamma(\frac{\alpha}{2}+1+n)}{\Gamma(\alpha+1+n)} &< (\alpha+n)^{\frac{-\alpha}{2}}, \end{aligned}$$

for $0 < \alpha < 1$ and we conclude that

$$k^{\alpha+1}(2n) < 2^\alpha k^{\alpha+1}(n)\left(1 + \frac{1-\alpha}{2(\alpha+n)}\right)^{\frac{\alpha}{2}} \le 2^\alpha k^{\alpha+1}(n)\left(1 + \frac{1-\alpha}{2(1+\alpha)}\right)^\alpha,$$

for $n \ge 1$ and $0 < \alpha < 1$. $\square$

The Cesàro sum of order α of f is defined by

$$\Delta^{-\alpha}f(n) := (k^\alpha * f)(n) = \sum_{j=0}^{n} k^\alpha(n-j)f(j), \qquad n \in \mathbb{N}_0, \alpha > 0.$$

Again we prefer to follow the notation $\Delta^{-\alpha}f(n)$ instead of $S_n^{\alpha-1}$ used in [29]. Note that $\Delta^{-\alpha-\beta}f = k^\beta * (\Delta^{-\alpha}f)$ and then $\Delta^{-\alpha}\Delta^{-\beta} = \Delta^{-(\alpha+\beta)} = \Delta^{-\beta}\Delta^{-\alpha}$ for $\alpha, \beta > 0$, for more details see again [29, Vol. I, p.76-77]. Note also that $\lim_{\alpha\to 0}\Delta^{-\alpha}f(n) = f(n)$ with $\alpha > 0$ and $n \in \mathbb{N}_0$.

We write $W = -\Delta$, $W^m = (-1)^m\Delta^m$ for $m \in \mathbb{N}$. The operator W has inverse in $c_{0,0}$, $W^{-1}f(n) = \sum_{j=n}^{\infty} f(j)$ and its iterations are given by the sum

$$W^{-m}f(n) = \sum_{j=m}^{\infty} \frac{\Gamma(j-n+m)}{\Gamma(j-n+1)\Gamma(m)} f(j) = \sum_{j=n}^{\infty} k^m(j-n)f(j), \qquad n \in \mathbb{N}_0$$

for each scalar-valued sequence f such that $\sum_{n=0}^{\infty} |f(n)|n^m < \infty$, see for example [17, p.307]. These facts and the clear connection with the Weyl fractional calculus motivates the following definition.

Definition 2.2. Let $f : \mathbb{N}_0 \to X$ and $\alpha > 0$ be given. The Weyl sum of order α of f, $W^{-\alpha}f$, is defined by

$$W^{-\alpha}f(n) := \sum_{j=n}^{\infty} k^\alpha(j-n)f(j), \qquad n \in \mathbb{N}_0,$$

whenever the right hand side makes sense. The Weyl difference of order α of f, $W^\alpha f$, is defined by

$$W^\alpha f(n) := W^m W^{-(m-\alpha)}f(n) = (-1)^m\Delta^m W^{-(m-\alpha)}f(n), \qquad n \in \mathbb{N}_0,$$

for $m = [\alpha] + 1$, whenever the right hand side makes sense. In particular $W^\alpha : c_{0,0} \to c_{0,0}$ for $\alpha \in \mathbb{R}$.

Observe that if $\alpha \in \mathbb{N}_0$, the Weyl difference of order α coincides with the definition given in [17, Section 4]. Some general properties are shown in the following proposition.

Proposition 2.3. *Let $f \in c_{0,0}(X)$. The following assertions hold:*

(i) *For $\alpha, \beta > 0$, $W^{-\alpha}W^{-\beta}f = W^{-(\alpha+\beta)}f = W^{-\beta}W^{-\alpha}f$.*
(ii) *For $\alpha > 0$ and $n \in \mathbb{N}_0$, we have $\lim_{\alpha\to 0^+} W^{-\alpha}f(n) = f(n)$.*
(iii) *For $\alpha > 0$, $W^\alpha W^{-\alpha}f = W^{-\alpha}W^\alpha f = f$.*
(iv) *For $\alpha > 0$ and $n \in \mathbb{N}_0$, we have $\lim_{\alpha\to 0^+} W^\alpha f(n) = f(n)$.*
(v) *For all $\alpha, \beta \in \mathbb{R}$ we have $W^\alpha W^\beta f = W^{\alpha+\beta}f = W^\beta W^\alpha f$.*

Proof. (i) It is clear using the Fubini theorem and the semigroup property $k^{\alpha+\beta} = k^\alpha * k^\beta$ for $\alpha, \beta > 0$. (ii) It is sufficient to apply that f has finite support and $\lim_{\alpha\to 0^+} k^\alpha(j) = e_0(j)$ for $j \in \mathbb{N}_0$. (iii) We write $m = [\alpha] + 1$. Applying part (i), for $n \in \mathbb{N}_0$, we have that

$$W^\alpha W^{-\alpha}f(n) = W^m W^{-(m-\alpha)}W^{-\alpha}f(n) = W^m W^{-m}f(n) = f(n),$$

since W^{-m} is the inverse of W^m in $c_{0,0}(X)$, see [17, Section 4]. On the other hand,

$$\begin{aligned} W^{-\alpha}W^{\alpha}f(n) &= W^{-(\alpha+1-m)}W^{-(m-1)}W^mW^{-(m-\alpha)}f(n) = W^{-(\alpha+1-m)}W^1W^{-(m-\alpha)}f(n) \\ &= W^{-(\alpha+1-m)}W^{-(m-\alpha)}f(n) - \sum_{j=n}^{\infty} k^{\alpha+1-m}(j-n)W^{-(m-\alpha)}f(j+1) \\ &= W^{-1}f(n) - \sum_{j=n+1}^{\infty} k^{\alpha+1-m}(j-n-1)W^{-(m-\alpha)}f(j) \\ &= W^{-1}f(n) - W^{-1}f(n+1) = f(n), \end{aligned}$$

where we use part (i).(iv) It is sufficient to apply that f has finite support and $\lim_{\alpha\to 0^+} k^{1-\alpha}(j) = 1$ for $j \in \mathbb{N}_0$. (v) It is simple to check using the previous results. $\square$

Example 2.4. (i) Let $\lambda \in \mathbb{C}\backslash\{0\}$, and $p_\lambda(n) := \lambda^{-(n+1)}$ for $n \in \mathbb{N}_0$. An easy computation shows that the sequence p_λ is a pseudo-resolvent, that is, it satisfies the Hilbert equation

$$(\mu - \lambda)(p_\lambda * p_\mu)(n) = p_\lambda(n) - p_\mu(n), \quad n \in \mathbb{N}_0.$$

Moreover, the following identity holds

$$p_\lambda * (\lambda e_0 - e_1) = e_0, \qquad \lambda \in \mathbb{C}\backslash\{0\}.$$

We claim that the functions p_λ are eigenfunctions for the operator W^α for $\alpha \in \mathbb{R}$ and $|\lambda| > 1$: we have, by (2.1), that

$$W^{-\alpha}p_\lambda(n) = \lambda^{-(n+1)}\sum_{j=0}^{\infty} k^\alpha(j)\lambda^{-j} = \frac{\lambda^\alpha}{(\lambda-1)^\alpha}p_\lambda(n), \qquad n \in \mathbb{N}_0.$$

By Proposition 2.3 (iii), we obtain that

$$W^\alpha p_\lambda = \frac{(\lambda-1)^\alpha}{\lambda^\alpha}p_\lambda, \qquad |\lambda| > 1.$$

(ii) Let $\alpha \geq 0$ and $n \in \mathbb{N}_0$ be given. We define

$$h_n^\alpha(j) := \begin{cases} k^\alpha(n-j), & j \leq n \\ 0, & j > n. \end{cases}$$

Functions h_n^α are denoted by $\Gamma_n^{\alpha-1}$ for $\alpha \in \mathbb{N}_0$ in [17, Section 4]. Note that $h_n^\alpha \in c_{0,0}$ for $n \in \mathbb{N}_0$, in fact, $h_n^\alpha \in \operatorname{span}\{e_j \mid 0 \leq j \leq n\}$, $h_0^\alpha = e_0$, $h_1^\alpha = \alpha e_0 + e_1$, $h_n^0 := \lim_{\alpha\to 0^+} h_n^\alpha = e_n$, and

$$h_n^\alpha(j) = k^\alpha(n-j) = \sum_{l=0}^{n} k^\alpha(n-l)e_l(j) = \sum_{l=0}^{n} k^\alpha(n-l)e_1^{*l}(j), \qquad 0 \leq j \leq n. \tag{2.4}$$

Then for all $\beta \geq 0$ it is easy to check that $W^{-\beta}h_n^\alpha = h_n^{\alpha+\beta}$, i.e.,

$$W^{-\beta}h_n^\alpha(j) = \sum_{i=j}^{\infty} k^\beta(i-j)h_n^\alpha(i) = h_n^{\alpha+\beta}(j), \qquad j \in \mathbb{N}_0.$$

Using Proposition 2.3 (iii), we obtain that

$$W^\beta h_n^\alpha(j) = h_n^{\alpha-\beta}(j), \qquad j \in \mathbb{N}_0,$$

for $0 \le \beta \le \alpha$ and $n \in \mathbb{N}_0$.

The following remark shows an interesting duality between the operator $\Delta^{-\alpha}$ and $W^{-\alpha}$. Similar results may be found in [1, Section 4] and [2, Theorem 4.1 and 4.4].

Remark 2.5. Let $f, g \in c_{0,0}$, we consider the usual duality product $\langle\, , \,\rangle$ given by

$$\langle f, g\rangle := \sum_{n=0}^{\infty} f(n)g(n).$$

By Fubini theorem, we get that $\langle W^{-\alpha}f, g\rangle = \langle f, \Delta^{-\alpha}g\rangle$ and consequently,

$$\langle f, g\rangle = \langle W^{\alpha}f, \Delta^{-\alpha}g\rangle = \langle \Delta^{-\alpha}f, W^{\alpha}g\rangle.$$

The next lemma includes a equality which is a important tool for further developments in this paper. The proof runs parallel to the proof of the integer case given in [17, Lemma 4.4] and we do not include here.

Lemma 2.6. *Let $f, g \in c_{0,0}$ and $\alpha \ge 0$, then*

$$\begin{aligned} W^{\alpha}(f * g)(n) &= \sum_{j=0}^{n} W^{\alpha}g(j) \sum_{p=n-j}^{n} k^{\alpha}(p-n+j)W^{\alpha}f(p) \\ &\quad - \sum_{j=n+1}^{\infty} W^{\alpha}g(j) \sum_{p=n+1}^{\infty} k^{\alpha}(p-n+j)W^{\alpha}f(p). \end{aligned}$$

Following definitions are inspired in [15, Definition 1.3].

Definition 2.7. Let $\alpha > 0$. We say that a positive sequence ϕ belongs to the class $\omega_{\alpha,loc}$, if there is a constant $c_\phi > 0$ such that

$$\left(\sum_{n=0}^{j} + \sum_{n=p+1}^{j+p}\right) k^{\alpha}(n)\phi(j+p-n) \le c_\phi \phi(j)\phi(p), \qquad 1 \le j \le p. \tag{2.5}$$

Moreover, we denote by ω_α the set of nondecreasing sequences $\phi \in \omega_{\alpha,loc}$ which are of exponential type and satisfy $\inf\limits_{n\ge 0}(k^{\alpha+1}(n))^{-1}\phi(n) > 0$

Example of sequences in ω_α are the following ones:

(i) any nondecreasing sequence ϕ satisfying $\max(k^{\alpha+1}(n), \phi(2n)) \le M\phi(n)$ for some $M > 0$ and for each $n \ge 0$ (in particular $\phi(n) = n^{\beta}(1+n^{\mu})$ with $\beta + \mu \ge \alpha$ and $\beta, \mu \ge 0$ and $\phi(n) = k^{\gamma}(n)$ with $\gamma \ge \alpha + 1$).

(ii) $\phi(n) = k^{\alpha+1}(n)\rho(n)$, where ρ is a nondecreasing weight, i.e., $\rho(n+m) \le C\rho(n)\rho(m)$ for $n, m \in \mathbb{N}_0$.

(iii) $\phi(n) = k^{\nu+1}(n)e^{\lambda n}$ for $\nu, \lambda > 0$.

By the equivalence $k^{\alpha}(n) \sim \dfrac{n^{\alpha-1}}{\Gamma(\alpha)}$, see formula (2.2), equivalent examples may be given in terms of $n^{\alpha-1}$. The particular case $\phi(n) = k^{\alpha+1}(n)$ will play a fundamental role in this paper, and the condition (2.5) is improved.

Lemma 2.8. *For $0<\alpha<1$, the following inequality holds*

$$\left(\sum_{n=0}^{j}+\sum_{n=p+1}^{j+p}\right)k^\alpha(n)k^{\alpha+1}(j+p-n)\le\left(2^{\alpha+1}\left(1+\frac{1-\alpha}{2(1+\alpha)}\right)^\alpha-1\right)k^{\alpha+1}(j)k^{\alpha+1}(p),\quad 1\le j\le p.$$

Proof. For $1\le j\le p$, and $\alpha>0$, we have that

$$\begin{aligned}\sum_{n=0}^{j}k^\alpha(n)k^{\alpha+1}(j+p-n) &\le k^{\alpha+1}(j+p)\sum_{n=0}^{j}k^\alpha(n)=k^{\alpha+1}(j+p)k^{\alpha+1}(j)\\ \sum_{n=p+1}^{j+p}k^\alpha(n)k^{\alpha+1}(j+p-n) &\le k^{\alpha+1}(j-1)\sum_{n=p+1}^{j+p}k^\alpha(n)\le k^{\alpha+1}(j)\left(k^{\alpha+1}(j+p)-k^{\alpha+1}(p)\right).\end{aligned}$$

As $k^{\alpha+1}$ is an increasing sequence, we have $k^{\alpha+1}(j+p)\le k^{\alpha+1}(2p)$ for $j\le p$ and we apply the Lemma 2.1 to conclude the proof. □

Proposition 2.9. *Take $0<\alpha\le\beta$ and $\phi\in\omega_{\alpha,loc}$.*

(i) *Then $\omega_{\beta,loc}\subset\omega_{\alpha,loc}$ and $\omega_\beta\subset\omega_\alpha$.*
(ii) *$(k^\alpha*\phi)(2n)\le c_\phi\phi^2(n)$ for $n\ge1$.*
(iii) *$k^\alpha(n)\le c_\phi\phi(n)\le a^n$ for $n\ge1$ and some $a>0$.*
(iv) *$k^{2\alpha}(2n)\le c\phi^2(n)$ for $n\in\mathbb{N}_0$ and $c>0$.*
(v) *$\phi(n+1)\le C\phi(n)$ for some $C>0$ independent of $n\ge1$.*
(vi) *$k^\beta\in\omega_{\alpha,loc}$ if and only if $\beta\ge\alpha+1$.*

Proof. (i) Since $k^\beta(n)\ge k^\alpha(n)$ for $n\in\mathbb{N}_0$, then $\omega_{\beta,loc}\subset\omega_{\alpha,loc}$ and $\omega_\beta\subset\omega_\alpha$ for $\beta\ge\alpha>0$. (ii) It is enough to take $j=p$ in (2.5) to obtain the inequality. (iii) By part (ii), we have that

$$k^\alpha(n)\phi(n)\le(k^\alpha*\phi)(2n)\le c_\phi\phi^2(n),\qquad n\ge1,$$

and we get the first inequality. For $n\ge1$, we apply the inequality (2.5) $n-1$ times to obtain that

$$c_\phi\phi(n)=c_\phi k^\alpha(0)\phi(n-1+1)\le c_\phi^2\phi(1)\phi(n-1)\le(c_\phi\phi(1))^n.$$

(iv) We combine parts (ii), (iii) and the semigroup property of kernels k^α to conclude that

$$c_\phi\phi^2(n)\ge(k^\alpha*\phi)(2n)\ge c'(k^\alpha*k^\alpha)(2n)=c'k^{2\alpha}(2n),\qquad n\in\mathbb{N}_0,$$

for some $c'>0$. (v) Take $j=1$ and $p=n\ge1$ in (2.5) to get

$$\phi(n+1)=k^\alpha(0)\phi(n+1)\le\sum_{m=0}^{1}k^\alpha(m)\phi(n+1-m)\le c_\phi\phi(1)\phi(n),\qquad n\ge1.$$

(vi) If $k^\beta\in\omega_{\alpha,loc}$ then we apply (2.2) and part (ii) to get

$$(k^\alpha*k^\beta)(2n)=k^{\alpha+\beta}(2n)\sim2^{\alpha+\beta-1}\frac{n^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\le c\frac{n^{2(\beta-1)}}{\Gamma^2(\beta)},\qquad n\ge1$$

and we conclude that $\beta\ge\alpha+1$. Note that $k^{\alpha+1}\in\omega_{\alpha,loc}$ and then $k^\beta\in\omega_{\alpha,loc}$ for $\beta\ge\alpha+1$ for part (i) and we conclude the proof. □

For $\alpha \geq 0$, and $\phi \in \omega_{\alpha,loc}$, we define the application $q_\phi : c_{0,0} \to [0,\infty)$ given by

$$q_\phi(f) := \sum_{n=0}^{\infty} \phi(n) |W^\alpha f(n)|, \qquad f \in c_{0,0}.$$

Note that for $\alpha = 0$ the above application corresponds to the usual norm in ℓ^1_ϕ. In the case $\phi = k^{\alpha+1}$, we write q_α instead of $q_{k^{\alpha+1}}$ and $q_0 = \| \ \|_1$ for $\alpha \geq 0$. By (2.2), the norm q_α is equivalent to the norm $\widetilde{q_\alpha}$ given by

$$\widetilde{q_\alpha}(f) := |f(0)| + \sum_{n=1}^{\infty} n^\alpha |W^\alpha f(n)|.$$

This expression was considered for the case $\alpha \in \mathbb{N}_0$ in [17, Definition 4.2].

Part of the following result extends [17, Theorem 4.5] and the proof is similar to the proof of [15, Proposition 1.4]. We include the proof to give a complete view of this result.

Theorem 2.10. *Let $\alpha > 0$ and $\phi \in \omega_{\alpha,loc}$. The application q_ϕ defines a norm in $c_{0,0}$ and*

$$q_\phi(f * g) \leq C_\phi \, q_\phi(f) \, q_\phi(g), \qquad f, g \in c_{0,0},$$

with $C_\phi > 0$ independent of f and g. We denote by $\tau^\alpha(\phi)$ the Banach algebra obtained as the completion of $c_{0,0}$ in the norm q_ϕ. In the case that $\phi \in \omega_\alpha$ then

(i) *the operator Δ is linear and bounded on $\tau^\alpha(\phi)$, $\Delta \in \mathcal{B}(\tau^\alpha(\phi))$.*
(ii) *$\tau^\alpha(\phi) \hookrightarrow \tau^\alpha(k^{\alpha+1}) \hookrightarrow \ell^1$, and $\lim_{\alpha \to 0^+} q_\alpha(f) = \|f\|_1$, for $f \in c_{0,0}$.*
(iii) *for $0 < \alpha < \beta$, $\tau^\beta(k^{\beta+1}) \hookrightarrow \tau^\alpha(k^{\alpha+1})$.*
(iv) *for $0 < \alpha < 1$,*

$$q_\alpha(f * g) \leq \left(2^{\alpha+1} \left(1 + \frac{1-\alpha}{2(1+\alpha)} \right)^\alpha - 1 \right) q_\alpha(f) \, q_\alpha(g), \qquad f, g \in \tau^\alpha(k^{\alpha+1}).$$

Proof. It is clear that q_α is a norm in $c_{0,0}$. Now, applying Lemma 2.6 we have

$$\begin{aligned}
q_\phi(f*g) &\leq \left(\sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{p=n-j}^{n} + \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \sum_{p=n+1}^{\infty} \right) \phi(n) k^\alpha(p-n+j) |W^\alpha g(j)| |W^\alpha f(p)| \\
&= \left(\sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \sum_{p=n-j}^{n} + \sum_{j=1}^{\infty} \sum_{n=0}^{j-1} \sum_{p=n+1}^{\infty} \right) \phi(n) k^\alpha(p-n+j) |W^\alpha g(j)| |W^\alpha f(p)| \\
&= \left(\sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{n=\max(j,p)}^{p+j} + \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} \sum_{n=0}^{\min(j,p)-1} \right) \phi(n) k^\alpha(p-n+j) |W^\alpha g(j)| |W^\alpha f(p)| \\
&\leq \phi(0) |W^\alpha g(0)| |W^\alpha f(0)| + c_\phi \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} \phi(j) \phi(p) |W^\alpha g(j)| |W^\alpha f(p)| \leq C_\phi \, q_\phi(f) \, q_\phi(g)
\end{aligned}$$

where we use Fubini's Theorem twice and the inequality (2.5) to show the first inequality.

Now take $\phi \in \omega_\alpha$. (i) It is clear that Δ is a linear operator and

$$q_\phi(\Delta(f)) = \sum_{n=0}^{\infty} \phi(n) |W^\alpha f(n) - W^\alpha f(n+1)| \leq q_\phi(f) + \sum_{n=1}^{\infty} \phi(n-1) |W^\alpha f(n)| \leq 2 q_\phi(f),$$

for $f \in \tau^\alpha(\phi)$. (ii) It is clear that $\tau^\alpha(\phi) \hookrightarrow \tau^\alpha(k^{\alpha+1}) \hookrightarrow \ell^1$. By the Monotone Convergence Theorem and Proposition 2.3 (ii),

$$\lim_{\alpha\to 0^+} q_\alpha(f) = \lim_{\alpha\to 0^+} \sum_{n=0}^{\infty} k^{\alpha+1}(n)|W^\alpha f(n)| = \sum_{n=0}^{\infty} |f(n)| = \|f\|_1, \qquad f \in c_{0,0}.$$

(iii) Let $f \in c_{0,0}$, and $0 < \alpha < \beta$, then

$$\begin{aligned} q_\alpha(f) &= \sum_{n=0}^{\infty} k^{\alpha+1}(n)|W^\alpha f(n)| = \sum_{n=0}^{\infty} k^{\alpha+1}(n)|\sum_{j=n}^{\infty} k^{\beta-\alpha}(j-n)W^\beta f(j)| \\ &\le \sum_{j=0}^{\infty} |W^\beta f(j)| \sum_{n=0}^{j} k^{\beta-\alpha}(j-n)k^{\alpha+1}(n) = \sum_{j=0}^{\infty} k^{\beta+1}(j)|W^\beta f(j)| = q_\beta(f), \end{aligned}$$

where we have applied Proposition 2.3 (v) and the semigroup property of k^α. (iv) This inequality follows from Lemma (2.8). □

Example 2.11. Note that sequence $(h_n^\alpha)_{n\in\mathbb{N}_0} \subset \tau^\alpha(\phi)$ with $\phi \in \omega_{\alpha,loc}$: By Example 2.4 (ii), $q_\phi(h_n^\alpha) = \phi(n)$ for $n \in \mathbb{N}_0$. Then the series $\sum_{n=0}^{\infty} W^\alpha f(n) h_n^\alpha$ converges on $\tau^\alpha(\phi)$ for every $f \in \tau^\alpha(\phi)$. By Proposition 2.9 (iii)

$$|f(m)| \le \sum_{n=m}^{\infty} k^\alpha(n-m)|W^\alpha(f)(n)| \le c_\phi \sum_{n=m}^{\infty} \phi(n)|W^\alpha(f)(n)| \le c_\phi q_\phi(f), \qquad m \in \mathbb{N}_0,$$

wherever k^α or ϕ is non-decreasing functions, i.e., for $\alpha \ge 1$ or $\phi \in \omega_\alpha$ for example. And then $f = \sum_{n=0}^{\infty} W^\alpha f(n) h_n^\alpha$ on $\tau^\alpha(\phi)$.

Take $\phi \in \omega_\alpha$ such that $\phi(n) \le Ca^n$ for $a > 1$. Then $p_\lambda \in \tau^\alpha(\phi)$ for $|\lambda| > a$, where sequences p_λ are defined in Example 2.4 (i), and

$$q_\phi(p_\lambda) \le C \frac{|\lambda - 1|^\alpha}{|\lambda|^\alpha(|\lambda| - a)}, \qquad |\lambda| > a.$$

In the particular case $\phi = k^\gamma$, then $p_\lambda \in \tau^\alpha(k^\gamma)$ for $|\lambda| > 1$ and $\gamma \ge \alpha + 1$,

$$q_{k^\gamma}(p_\lambda) = \frac{|\lambda - 1|^\alpha |\lambda|^{\gamma-\alpha-1}}{(|\lambda| - 1)^\gamma}, \qquad |\lambda| > 1, \tag{2.6}$$

where we have applied Example 2.4 (i) and the formula (2.1).

3. Cesàro sums and algebra homomorphisms

In this section we present our main results. The algebra structure of Cesàro sums are presented in several ways: functional equation (Theorem 3.3), algebra homomorphism (Theorem 3.5) and resolvent operators (Theorem 4.4). Note that these approach in fact characterizes the growth of Cesàro sums, as Corollary 3.6 and Corollary 3.7 for (C,α)-bounded operators show. We recall the following definition.

Definition 3.1. Given a bounded operator $T \in \mathcal{B}(X)$, the Cesàro sum of order $\alpha > 0$ of T, $(\Delta^{-\alpha}\mathcal{T}(n))_{n\geq 0} \subset \mathcal{B}(X)$, is defined by

$$\Delta^{-\alpha}\mathcal{T}(n)x := (k^\alpha * \mathcal{T})(n)x = \sum_{j=0}^{n} k^\alpha(n-j)T^j x, \qquad x \in X; \quad n \in \mathbb{N}_0.$$

Note that we keep the notation $\mathcal{T}(n) = T^n$ for $n \in \mathbb{N}_0$.

Example 3.2. The canonical example of a family of Cesàro sum of order α in Banach algebras $\tau^\alpha(\phi)$ (in particular in ℓ^1) is the family $\{h_n^\alpha\}_{n\in\mathbb{N}_0}$ given in Example 2.4(ii). Note that $(h_n^\alpha)_{n\in\mathbb{N}_0} \subset \tau^\alpha(\phi)$ with $\phi \in \omega_{\alpha,loc}$, see Example 2.11. We write $\mathcal{E}(n) = e_1^{*n}$ to get $h_n^\alpha = \Delta^{-\alpha}\mathcal{E}(n)$ for $n \in \mathbb{N}_0$ by equation (2.4).

The following theorem characterizes sequences of operators which are Cesàro sums of some order $\alpha > 0$ and a fixed operator T.

Theorem 3.3. *Let $\alpha > 0$ and $T, (T_n)_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. Then the following assertions are equivalent.*

(i) $T_n = \Delta^{-\alpha}\mathcal{T}(n)$ *for* $n \in \mathbb{N}_0$.

(ii) $T_0 = I$ *and the following functional equation holds:*

$$T_nT_m = \sum_{u=m}^{n+m} k^\alpha(n+m-u)T_u - \sum_{u=0}^{n-1} k^\alpha(n+m-u)T_u \qquad n \geq 1,\, m \in \mathbb{N}_0. \tag{3.1}$$

Proof. Assume (i). It is clear $T_0 = I$ and we claim the identity (3.1). Take $n \geq 1$, $m \geq 0$, then

$$\begin{aligned}
T_nT_m &= \sum_{j=0}^{n}\sum_{i=0}^{m} k^\alpha(n-j)k^\alpha(m-i)T^{j+i} = \sum_{j=0}^{n}\sum_{u=j}^{m+j} k^\alpha(n-j)k^\alpha(m+j-u)T^u \\
&= \sum_{j=0}^{n}\sum_{u=0}^{m+j} k^\alpha(n-j)k^\alpha(m+j-u)T^u - \sum_{j=1}^{n}\sum_{u=0}^{j-1} k^\alpha(n-j)k^\alpha(m+j-u)T^u \\
&= \sum_{j=0}^{n} k^\alpha(n-j)T_{m+j} - \sum_{j=1}^{n}\sum_{u=0}^{j-1} k^\alpha(n-j)k^\alpha(m+j-u)T^u.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{u=0}^{j-1} k^\alpha(n-j)k^\alpha(m+j-u)T^u = \sum_{u=0}^{n-1}\sum_{j=u+1}^{n} k^\alpha(n-j)k^\alpha(m+j-u)T^u \\
&= \sum_{u=0}^{n-1}\sum_{l=u}^{n-1} k^\alpha(l-u)k^\alpha(m+n-l)T^u = \sum_{l=0}^{n-1} k^\alpha(m+n-l)\sum_{u=0}^{l} k^\alpha(l-u)T^u \\
&= \sum_{l=0}^{n-1} k^\alpha(m+n-l)T_l.
\end{aligned}$$

and the equality (3.1) is proven. Conversely, assume (ii). Define $T := T_1 - \alpha I$ and

$$S_n := \sum_{j=0}^{n} k^\alpha(n-j)T^j, \quad n \in \mathbb{N}_0.$$

It is clear that $S_0 = I = T_0$, and $S_1 = \alpha I + T = T_1$. Inductively, we suppose that $S_n = T_n$. Then using that S_n satisfies (3.1), we have that

$$S_{n+1} + k^\alpha(1)S_n - k^\alpha(n+1)I = S_nS_1 = T_nT_1 = T_{n+1} + k^\alpha(1)S_n - k^\alpha(n+1)I.$$

Then we conclude that $T_{n+1} = S_{n+1}$, and consequently $T_n = \Delta^{-\alpha}\mathcal{T}(n)$ for all $n \in \mathbb{N}_0$. $\square$

Remark 3.4. Given $\{T_n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ a sequence of bounded operators which verify the equality (3.1). Then the operator defined by $T := T_1 - \alpha I$ is called the generator of $\{T_n\}_{n\in\mathbb{N}_0}$. By Theorem 3.3, $T_n = \Delta^{-\alpha}\mathcal{T}(n)$ where $\mathcal{T}(n) = T^n$ for $n \in \mathbb{N}_0$. In particular, note that $\{h_n^\alpha\}_{n\in\mathbb{N}_0}$ satisfies (3.1) in $\tau^\alpha(\phi)$, see Example 3.2, and the generator is the element e_1.

The following is one of the main results of this paper.

Theorem 3.5. *Let $\alpha > 0$ and $T \in \mathcal{B}(X)$ such that $\|\Delta^{-\alpha}\mathcal{T}(n)\| \le C\phi(n)$ for $n \in \mathbb{N}_0$ with $\phi \in \omega_{\alpha,loc}$ and $C > 0$. Then there exists a bounded algebra homomorphism $\theta : \tau^\alpha(\phi) \to \mathcal{B}(X)$ given by*

$$\theta(f)x := \sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x, \qquad x \in X, \quad f \in \tau^\alpha(\phi).$$

Furthermore, the following identities hold.

(i) *For $n \in \mathbb{N}_0$, $\theta(h_n^\alpha) = \Delta^{-\alpha}\mathcal{T}(n)$, in particular $\theta(e_0) = I$ and $\theta(e_1) = T$.*
(ii) *For $f \in \tau^\alpha(\phi)$ such that $\Delta f \in \tau^\alpha(\phi)$ and $x \in X$, $T\theta(\Delta f)x = (I - T)\theta(f)x - f(0)x$.*
(iii) *In the case that $\sup_{n\in\mathbb{N}_0} \dfrac{(k^{\beta-\alpha} * \phi)(n)}{\psi(n)} < \infty$, for $0 < \alpha < \beta$ and $\psi \in \omega_{\beta,loc}$, then $\tau^\beta(\psi) \hookrightarrow \tau^\alpha(\phi)$ and*

$$\theta(f)x = \sum_{n=0}^{\infty} W^\beta f(n)\Delta^{-\beta}\mathcal{T}(n)x, \qquad x \in X, \quad f \in \tau^\beta(\psi).$$

(iv) *If $\|T\| \le a$ for some $a > 0$, then $\theta(f)x = \sum_{n=0}^{\infty} f(n)T^n(x)$, for $f \in \tau^\alpha(\phi) \cap \ell^1_{a^n}$, in particular $\theta(p_\lambda) = (\lambda - T)^{-1}$ for $|\lambda| > a$.*

Proof. Note that the map θ is well-defined, lineal and continuous, $\|\theta(f)x\| \le Cq_\alpha(f)\|x\|$, for $f \in \tau^\alpha(\phi)$ and $x \in X$. To see that θ is algebra homomorphism is sufficient to prove that $\theta(f * g) = \theta(f)\theta(g)$ for $f, g \in c_{0,0}$. By Lemma 2.6, we get that

$$\begin{aligned}
\theta(f*g)x &= \sum_{n=0}^{\infty} W^\alpha(f*g)(n)\Delta^{-\alpha}\mathcal{T}(n)x \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} W^\alpha g(j) \sum_{p=n-j}^{n} k^\alpha(p-n+j)W^\alpha f(p)\Delta^{-\alpha}\mathcal{T}(n)x \\
&\quad -\sum_{n=0}^{\infty}\sum_{j=n+1}^{\infty} W^\alpha g(j) \sum_{p=n+1}^{\infty} k^\alpha(p-n+j)W^\alpha f(p)\Delta^{-\alpha}\mathcal{T}(n)x.
\end{aligned}$$

We apply Fubini theorem to get that

$$\begin{aligned}
\theta(f*g)x &= \sum_{j=0}^{\infty} W^\alpha g(j)\sum_{p=0}^{j} W^\alpha f(p)\sum_{n=j}^{p+j} k^\alpha(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x \\
&\quad+\sum_{j=0}^{\infty} W^\alpha g(j)\sum_{p=j+1}^{\infty} W^\alpha f(p)\sum_{n=p}^{p+j} k^\alpha(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x \\
&\quad-\sum_{j=1}^{\infty} W^\alpha g(j)\sum_{p=1}^{j} W^\alpha f(p)\sum_{n=0}^{p-1} k^\alpha(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x \\
&\quad-\sum_{j=1}^{\infty} W^\alpha g(j)\sum_{p=j+1}^{\infty} W^\alpha f(p)\sum_{n=0}^{j-1} k^\alpha(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x \\
&= \sum_{j=1}^{\infty} W^\alpha g(j)\sum_{p=1}^{j} W^\alpha f(p)\Big(\sum_{n=j}^{p+j}-\sum_{n=0}^{p-1}\Big)k^\alpha(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x+W^\alpha g(0)W^\alpha f(0)x \\
&\quad+\sum_{j=0}^{\infty} W^\alpha g(j)\sum_{p=j+1}^{\infty} W^\alpha f(p)\Big(\sum_{n=p}^{p+j}-\sum_{n=0}^{j-1}\Big)k^\alpha(p-n+j)\Delta^{-\alpha}\mathcal{T}(n)x \\
&= \sum_{j=1}^{\infty} W^\alpha g(j)\sum_{p=1}^{j} W^\alpha f(p)\Delta^{-\alpha}\mathcal{T}(p)\Delta^{-\alpha}\mathcal{T}(j)x+W^\alpha g(0)W^\alpha f(0)x \\
&\quad+\sum_{j=0}^{\infty} W^\alpha g(j)\sum_{p=j+1}^{\infty} W^\alpha f(p)\Delta^{-\alpha}\mathcal{T}(p)\Delta^{-\alpha}\mathcal{T}(j)x=\theta(f)\theta(g)x.
\end{aligned}$$

where we have used the identity (3.1).

(i) Note that $W^\alpha h_n^\alpha = e_n$, see Example 2.4 (ii), and then $\theta(h_n^\alpha)=\Delta^{-\alpha}\mathcal{T}(n)$ for $n\in\mathbb{N}_0$. As $e_0=h_0$ and $e_1=h_1^\alpha-\alpha h_0^\alpha$, it is clear that $\theta(e_0)=I$ and $\theta(e_1)=T$. (ii) Now, for $f\in\tau^\alpha(\phi)$ such that $\Delta f\in\tau^\alpha(\phi)$ and $x\in X$, we have that

$$\begin{aligned}
T\theta(\Delta f)x &= T\Big(\sum_{n=0}^{\infty} W^\alpha f(n+1)\Delta^{-\alpha}\mathcal{T}(n)x-\sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x\Big) \\
&= \sum_{n=0}^{\infty} W^\alpha f(n+1)\left(\Delta^{-\alpha}\mathcal{T}(n+1)x-k^\alpha(n+1)x\right)-T\sum_{n=0}^{\infty} W^\alpha f(n)\Delta^{-\alpha}\mathcal{T}(n)x \\
&= (I-T)\theta(f)x-W^\alpha f(0)\Delta^{-\alpha}\mathcal{T}(0)x-\sum_{n=0}^{\infty} W^\alpha f(n+1)k^\alpha(n+1)x \\
&= (I-T)\theta(f)x-\sum_{n=0}^{\infty} W^\alpha f(n)k^\alpha(n)x=(I-T)\theta(f)x-f(0)x,
\end{aligned}$$

where we have applied that $T\Delta^{-\alpha}\mathcal{T}(n) = \Delta^{-\alpha}\mathcal{T}(n+1) - k^{\alpha}(n+1)$ and $\displaystyle\sum_{n=0}^{\infty} W^{\alpha} f(n) k^{\alpha}(n) = f(0)$ for $f \in \tau(\phi)$. (iii) Suppose that $\displaystyle\sup_{n\in\mathbb{N}_0} \frac{(k^{\beta-\alpha} * \phi)(n)}{\psi(n)} < \infty$, with $0 < \alpha < \beta$ and $\psi \in \omega_{\beta,loc}$, then it is straightforward to check that $\tau^{\beta}(\psi) \hookrightarrow \tau^{\alpha}(\phi)$ and

$$\sum_{n=0}^{\infty} W^{\alpha} f(n)\Delta^{-\alpha}\mathcal{T}(n)x = \sum_{n=0}^{\infty} W^{\beta} f(n)\Delta^{-\beta}\mathcal{T}(n)x, \qquad f \in \tau^{\beta}(\psi),\ x \in X,$$

where we have applied Proposition 2.3 (v) and Remark 2.5. (iv) Now take $a > 0$ such that $\|T\| \le a$ and then $\sigma(T) \subset \{z \in \mathbb{C} \mid |z| \le a\}$. For $f \in \tau^{\alpha}(\phi) \cap \ell^1_{a^n}$, we apply Remark 2.5 to get

$$\theta(f)x = \sum_{n=0}^{\infty} f(n) T^n(x), \qquad x \in X.$$

In particular $p_{\lambda} \in \tau^{\alpha}(\phi) \cap \ell(a^n)$ for $|\lambda| > a$ and $\theta(p_{\lambda})x = \dfrac{1}{\lambda}\displaystyle\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} x = (\lambda - T)^{-1}x$ for $x \in X$. $\square$

Corollary 3.6. *Let $\alpha > 0$, $\phi \in \omega_{\alpha}$ and $\theta : \tau^{\alpha}(\phi) \to \mathcal{B}(X)$ be an algebra homomorphism. Then there exists $T \in \mathcal{B}(X)$ such that*

$$\theta(f)x = \sum_{n=0}^{\infty} W^{\alpha} f(n)\Delta^{-\alpha}\mathcal{T}(n)x, \qquad f \in \tau^{\alpha}(\phi), \quad x \in X;$$

in particular $\theta(h_n^{\alpha}) = \Delta^{-\alpha}\mathcal{T}(n)$ for $n \in \mathbb{N}_0$ and $\theta(p_{\lambda}) = (\lambda - T)^{-1}$ for $|\lambda| > \|T\|$.

Proof. Take $T := \theta(e_1)$. Note that $e_1 = h_1^{\alpha} - \alpha h_0^{\alpha}$, see Example 2.4 (ii), and $h_n^{\alpha} = \Delta^{-\alpha}\mathcal{E}(n)$ for $n \in \mathbb{N}_0$ where $\mathcal{E}(n) = e_1^{*n}$, see Example 3.2. By Example 2.11, $f = \displaystyle\sum_{j=0}^{\infty} W^{\alpha} f(n) h_n^{\alpha}$ for $f \in \tau^{\alpha}(\phi)$, we apply the continuity of θ to get

$$\begin{aligned} \theta(h_n^{\alpha})x &= \sum_{j=0}^{n} k^{\alpha}(n-j)\,(\theta(e_1))^j\, x = \Delta^{-\alpha}\mathcal{T}(n)x; \\ \theta(f)x &= \sum_{n=0}^{\infty} W^{\alpha} f(n)\theta(h_n^{\alpha})x = \sum_{n=0}^{\infty} W^{\alpha} f(n)\Delta^{-\alpha}\mathcal{T}(n)x, \end{aligned}$$

for $x \in X$. By Theorem 3.5 (iv), we conclude the proof. $\square$

By Theorem 3.5 and Corollary 3.6, we obtain the following characterizations of (C, α)-bounded and power-bounded operators.

Corollary 3.7. *Let $T \in \mathcal{B}(X)$ and $\alpha > 0$ be given. The following assertions are equivalent:*

(i) *T is (C, α)-bounded operator.*

(ii) *There exists a bounded algebra homomorphism $\theta : \tau^{\alpha}(k^{\alpha+1}) \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.*

In the limit case, the following assertions are equivalent:

(a) *T is power bounded.*

(b) *There exists a bounded algebra homomorphism $\theta : \ell^1 \to \mathcal{B}(X)$ such that $\theta(e_1) = T$.*

(c) *For any* $0 < \alpha < 1$, *there exist bounded algebra homomorphisms* $\theta_\alpha : \tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ *such that* $\theta_\alpha(e_1) = T$ *and* $\sup_{0<\alpha<1} \|\theta_\alpha\| < \infty$.

Proof. Due to previous results, we only have to check that (c) implies (b). Since the map θ_α is an algebra homomorphism then $\theta_\alpha(e_n) = T^n$, $\theta_\alpha(f)$ is well defined for $f \in c_{0,0}$ and is independent of α. Take $C > 0$ such that $\sup_{0<\alpha<1} \|\theta_\alpha\| < C$. We define $\theta(f) := \theta_\alpha(f)$ for $f \in c_{0,0}$ and some given $\alpha \in (0,1)$. Then $\|\theta(f)\| = \|\theta_\alpha(f)\| \le C\, q_\alpha(f)$ for $f \in c_{0,0}$. By Theorem 2.10 (ii), we get that $\|\theta(f)\| \le C\|f\|_1$, for $f \in c_{0,0}$ and we conclude the result by density. $\square$

4. The Z-transform and resolvent operators

Let $f : \mathbb{N}_0 \to X$ be a scalar sequence on a Banach space X. We also recall that the Z-transform of a given sequence $f : \mathbb{N}_0 \to X$ is defined by

$$\tilde{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}, \tag{4.1}$$

for all z such that this series converges. The set of numbers z in the complex plane for which series (4.1) converges is called the region of convergence of $\tilde{f}$. The uniqueness of the inverse Z-transform may be established as follows: suppose that there are two sequences f, and g with the same Z-transform, that is,

$$\sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} g(n) z^{-n}, \qquad |z| > R.$$

It follows from Laurent's theorem that $f(n) = g(n)$ for $n \in \mathbb{N}_0$.

Let $\phi : \mathbb{N}_0 \to (0,\infty)$ be a sequence such that $\phi(n) \le Ca^n$ for some $C > 0$ and $a > 0$. To follow the notation given in [8], we write $\omega = \log(a)$ and ω is a bound for the counting measure supported on $\mathbb{N}_0$, i.e., $\epsilon_\lambda \in \ell^1_\phi$ for $\lambda > \omega$ where $\epsilon_\lambda(n) := e^{-\lambda n}$ and $n \in \mathbb{N}_0$. Let $C^\infty((\omega,\infty), X)$ be the space of X-valued functions on (ω,∞) infinitely differentiable in the norm topology of X. For $r \in C^\infty((\omega,\infty), X)$, set

$$\|r\|_{W,\phi,\omega} := \sup\{ \frac{\|r(\lambda)\|}{\|\beta_{k,\lambda}\|_{1,\phi}} \mid k \in \mathbb{N}_0, \lambda > \omega\},$$

where $\beta_{k,\lambda}(n) = n^k e^{-\lambda n}$ for $n \in \mathbb{N}_0$ and $\lambda > \omega$. The Widder space $C^\infty_W((\omega,\infty), X; \phi)$ is defined by

$$C^\infty_W((\omega,\infty), X; \phi) = \{ r \in C^\infty((\omega,\infty), X) \mid \|r\|_{W,\phi,\omega} < \infty\}.$$

Endowed with the norm $\|\cdot\|_{W,\phi,\omega}$, the space $C^\infty_W((\omega,\infty), X; \phi)$ is a Banach space, see more details in [8, Section 1]. A direct consequence of [8, Theorem 1.2] is the following result.

Theorem 4.1. *Let* $\phi : \mathbb{N}_0 \to (0,\infty)$ *be a sequence such that* $\phi(n) \le Ca^n$ *for some* $C > 0$ *and* $a > 0$. *Take now* $f : \mathbb{N}_0 \to X$ *a vector-valued sequence. Then the following assertions are equivalent.*

(i) $\sup_{n\in\mathbb{N}_0} \dfrac{\|f(n)\|}{\phi(n)} < \infty$.

(ii) *There exists* $\theta : \ell^1_\phi \to X$ *such that* $\theta(\lambda p_\lambda) = \tilde{f}(\lambda)$ *for* $\lambda > a$.

(iii) $\tilde{f} \circ \exp \in C^\infty_W((\log(a),\infty), X; \phi)$.

Proof. To show that (i) implies (ii), we define $\theta(g) := \sum_{n=0}^{\infty} g(n)f(n)$ for $g = (g(n))_{n\geq 0} \in \ell^1_\phi$. Now consider the part (ii). We define $h(n) := \theta(e_n)$ for $n \in \mathbb{N}_0$. It is clear that $\sup_{n\in\mathbb{N}_0} \dfrac{\|h(n)\|}{\phi(n)} < \infty$ and

$$\tilde{f}(\lambda) = \theta(\lambda p_\lambda) = \sum_{n\in\mathbb{N}_0} \theta(e_n)\lambda^n = \tilde{h}(\lambda), \qquad |\lambda| > a,$$

where we conclude that $h(n) = f(n)$ for $n \in \mathbb{N}_0$ and part (i) is proved. Now take again part (ii). Due to [8, Theorem 1.2],

$$\theta(\epsilon_\mu) = \theta(\exp(\mu)p_{\exp(\mu)}) = (\tilde{f} \circ \exp)(\mu), \qquad \mu > \log(a),$$

and we conclude the part (iii). Finally suppose that $\tilde{f}\circ\exp \in C^\infty_W((\log(a), \infty), X; \phi)$. Again by [8, Theorem 1.2], there exists a bounded homomorphism $\theta : \ell^1_\phi \to X$ such that $\theta(\epsilon_\mu) = (\tilde{f} \circ \exp)(\mu)$ for $\mu > \log(a)$. Since $\epsilon_\mu(n) = e^{-\mu n} = e^\mu p_{e^\mu}(n)$, we conclude that $\theta(\lambda p_\lambda) = \tilde{f}(\lambda)$ for $\lambda > a$. ☐

Remark 4.2. Note that Theorem 4.1 is closely connected to [8, Theorem 4.2], where the Banach space X has the Radon-Nikodym property, RNP, to may identity the Widder space $C^\infty_W((\omega,\infty), X; \mathrm{m})$ and $L^\infty(\mathbb{R}_+, X; \mathrm{m})$. The RNP is a well-known property in the theory of function spaces. This property passes to closed subspaces (hereditary property) and is enjoyed by any reflexive space, any separable dual space, and any $\ell^1(\Gamma)$ space, where Γ is a set, see definitions and more details in [3, Section 1.2].

In the well-known scalar version, $X = \mathbb{C}$, the following Z-transforms are obtained directly:

$$\begin{aligned}
\widetilde{e_n}(z) &= z^{-n}, \qquad z \neq 0, \quad n \in \mathbb{N}_0;\\
\widetilde{k^\alpha}(z) &= \frac{z^\alpha}{(z-1)^\alpha}, \qquad |z| > 1;\\
\widetilde{p_\lambda}(z) &= \frac{z}{z\lambda - 1}, \qquad |z| > \frac{1}{|\lambda|}, \ \lambda \in \mathbb{C}\backslash\{0\},\\
\widetilde{h^\alpha_n}(z) &= \sum_{j=0}^{n} k^\alpha(n-j)z^{-j}, \qquad z \neq 0.
\end{aligned}$$

It is also well-known that

$$\widetilde{(f * g)}(z) = \widetilde{f}(z)\widetilde{g}(z), \tag{4.2}$$

wherever these Z-transforms converge on $z \in \mathbb{C}$, see these results and many other properties of the Z-transform in, for example [12, Chapter 6]. In particular, given $\alpha > 0$ and $f : \mathbb{N}_0 \to X$ such that $\tilde{f}(z)$ converges for $|z| > R$, then

$$(\widetilde{\Delta^{-\alpha} f})(z) = \frac{z^\alpha}{(z-1)^\alpha}\widetilde{f}(z), \qquad |z| > \max\{R, 1\}.$$

We denote by ${}_nf(m) := f(n+m)$ for all $m, n \in \mathbb{N}_0$. Next technical lemma for the Z-transform is applied in Theorem 4.4. Similar results hold for the Laplace transform, see for example [23, Proposition 4.1].

Lemma 4.3. *Let X be a Banach space, $f:\mathbb{N}_0\to\mathbb{C}$ a scalar sequence and $S:\mathbb{N}_0\to\mathcal{B}(X)$ a vector-operator valued sequence. Then*

$$\begin{aligned}\frac{1}{\mu-\lambda}\widetilde{f}(\mu)\Big(\mu\widetilde{S}(\lambda)x-\lambda\widetilde{S}(\mu)x\Big) &= \sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}(f*{}_nS)(m)x,\\ \frac{1}{\mu-\lambda}\Big(\mu\widetilde{f}(\lambda)-\lambda\widetilde{f}(\mu)\Big)\widetilde{S}(\mu)x &= \sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}({}_nf*S)(m)x,\end{aligned}$$

for $|\lambda|>|\mu|$ sufficiently large where the double Z-transform converge and $x\in X$.

Proof. To show the first equality, note that,

$$\widetilde{{}_nS}(\mu)x=\sum_{m=0}^{\infty}\mu^{-m}S(m+n)x=\mu^n\sum_{j=n}^{\infty}\mu^{-j}S(j)x=\mu^n\Big(\widetilde{S}(\mu)x-\sum_{j=0}^{n-1}\mu^{-j}S(j)x\Big),$$

for $x\in X$ and $n\geq 1$. Then we get that

$$\begin{aligned}&\sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}(f*{}_nS)(m)x=\widetilde{f}(\mu)\sum_{n=0}^{\infty}\lambda^{-n}\widetilde{{}_nS}(\mu)x=\widetilde{f}(\mu)\Big(\widetilde{S}(\mu)x+\sum_{n=1}^{\infty}\lambda^{-n}\widetilde{{}_nS}(\mu)x\Big)\\ &=\widetilde{f}(\mu)\widetilde{S}(\mu)x\sum_{n=0}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n-\widetilde{f}(\mu)\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n\sum_{j=0}^{n-1}\mu^{-j}S(j)x.\end{aligned}$$

where we have applied the equality (4.2). Finally, as

$$\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n\sum_{j=0}^{n-1}\mu^{-j}S(j)x=\sum_{j=0}^{\infty}\mu^{-j}S(j)x\sum_{n=j+1}^{\infty}\Big(\frac{\mu}{\lambda}\Big)^n=\frac{\mu}{\lambda-\mu}\widetilde{S}(\lambda)x,$$

we conclude that

$$\sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}(f*{}_nS)(m)x=\frac{1}{\lambda-\mu}\widetilde{f}(\mu)\Big(\lambda\widetilde{S}(\mu)x-\mu\widetilde{S}(\lambda)x\Big),$$

for $|\lambda|>|\mu|$ sufficiently large and $x\in X$. Following these ideas, the second equality is also shown. ☐

Theorem 4.4. *Let $\alpha\geq 0$, X a Banach space, $\{T_n\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$ such that $T_0=I$, $\|T_n\|\leq C\phi(n)\leq C'a^n$ ($\phi\in\omega_\alpha$ and $a>1$) for all $n\in\mathbb{N}_0$ with $C,C'>0$. The following statements are equivalent:*

(i) *The operator-valued sequence $\{T_n\}_{n\in\mathbb{N}_0}$ satisfies the equation* (3.1).
(ii) *There exists a bounded algebra homomorphism $\theta:\tau^\alpha(\phi)\to\mathcal{B}(X)$ such that $\theta(h_n^\alpha)=T_n$ for $n\in\mathbb{N}_0$.*
(iii) *The family $\{R(\lambda)\}_{|\lambda|>a}$ defined by*

$$R(\lambda)x:=\frac{(\lambda-1)^\alpha}{\lambda^{\alpha+1}}\sum_{n=0}^{\infty}\lambda^{-n}T_n(x),\qquad |\lambda|>a,\ x\in X,$$

is a pseudo-resolvent.

In these cases the generator of $\{T_n\}_{n\in\mathbb{N}_0}$*, defined by* $T := T_1 - \alpha I$ *in Remark 3.4, satisfies that* $T_n = \Delta^{-\alpha}\mathcal{T}(n)$ *for* $n \in \mathbb{N}_0$*,* $\theta(e_1) = T$*,* $\{\lambda \in \mathbb{C} \mid |\lambda| > a\} \subset \rho(T)$ *and*

$$R(\lambda) = (\lambda - T)^{-1}, \qquad |\lambda| > a.$$

Proof. The proof (i)⇒(ii) is a direct consequence of Theorem 3.3 and Theorem 3.5. To show that (ii)⇒(iii), we use that Corollary 3.6. Finally we prove (iii)⇒(i). It is clear that

$$R(\lambda) = \frac{\widetilde{\mathfrak{T}}(\lambda)}{\lambda\widetilde{k^\alpha}(\lambda)}, \qquad |\lambda| > a,$$

where $\mathfrak{T} = \{T_n\}_{n\in\mathbb{N}_0}$ and $\widetilde{\mathfrak{T}}$ is given by (4.1). Since $\{R(\lambda)\}_{|\lambda|>a}$ is a pseudo-resolvent, then

$$(\mu - \lambda)\frac{\widetilde{\mathfrak{T}}(\lambda)\widetilde{\mathfrak{T}}(\mu)}{\lambda\widetilde{k^\alpha}(\lambda)\mu\widetilde{k^\alpha}(\mu)} = \frac{\widetilde{\mathfrak{T}}(\lambda)}{\lambda\widetilde{k^\alpha}(\lambda)} - \frac{\widetilde{\mathfrak{T}}(\mu)}{\mu\widetilde{k^\alpha}(\mu)}, \qquad |\lambda|, |\mu| > a, \quad \mu \neq \lambda,$$

so

$$\widetilde{\mathfrak{T}}(\lambda)\widetilde{\mathfrak{T}}(\mu) = \frac{1}{\mu - \lambda}\Big(\mu\widetilde{k^\alpha}(\mu)\widetilde{\mathfrak{T}}(\lambda) - \lambda\widetilde{k^\alpha}(\lambda)\widetilde{\mathfrak{T}}(\mu)\Big), \qquad |\lambda|, |\mu| > a, \quad \mu \neq \lambda.$$

On the other hand, note that the condition (3.1) is expressed by

$$(k^\alpha * {}_n\mathfrak{T})(m) - ({}_nk^\alpha * \mathfrak{T})(m) + k^\alpha(n)T_m = \sum_{u=n}^{n+m} k^\alpha(n+m-u)T_u - \sum_{u=0}^{m-1} k^\alpha(n+m-u)T_u,$$

for $m \geq 1$ and $n \geq 0$. We apply Lemma 4.3 and do some simple operations to get that

$$\sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{\infty}\mu^{-m}\left((k^\alpha * {}_n\mathfrak{T})(m) - ({}_nk^\alpha * \mathfrak{T})(m) + k^\alpha(n)T_m\right) = \frac{\mu\widetilde{k^\alpha}(\mu)\widetilde{\mathfrak{T}}(\lambda) - \lambda\widetilde{k^\alpha}(\lambda)\widetilde{\mathfrak{T}}(\mu)}{\mu - \lambda},$$

for $|\lambda|, |\mu| > a$, and $\mu \neq \lambda$. Then we conclude that $\{T_n\}_{n\in\mathbb{N}_0}$ satisfies (3.1), as consequence of the injectivity of the double Z-transform. Finally, by Corollary 3.6

$$R(\lambda) = \theta(p_\lambda) = (\lambda - T)^{-1}, \qquad |\lambda| > a,$$

and we finish the proof. □

5. Applications, examples and final comments

In this last section, we present some applications, comments, examples and counterexamples of some results presented in this paper.

5.1. **Bounds for Abel means.** Given $T \in \mathcal{B}(X)$ and $0 \leq r < 1$ the Abel mean of order r of operator T, $A_r(T)$, is defined by

$$A_r(T)x := (1 - r)\sum_{n=0}^{\infty} r^n T^n(x), \qquad x \in X,$$

when this series converges for some $r \in [0, 1)$, see for example [25]. Note that for $0 < r < \frac{1}{r(T)}$ then $\frac{1}{r} \in \rho(T)$ and

$$A_r(T) = \frac{(1 - r)}{r}\left(\frac{1}{r} - T\right)^{-1}, \qquad 0 < r < \min\{1, \frac{1}{r(T)}\},$$

where $r(T) = \lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}$ denotes the spectral radius of T.

The next theorem improves [25, Proposition 2.1 (i)] given for $\alpha \in \{0,1\}$.

Theorem 5.1. *Take $\alpha \geq 0$ and $T \in \mathcal{B}(X)$. Then*

$$A_r(T)x = (1-r)^{\alpha+1}\sum_{n=0}^{\infty} r^n \Delta^{-\alpha}\mathcal{T}(n)x, \qquad 0 \leq r < \min\{1, \frac{1}{r(T)}\}.$$

In the case that $\|\Delta^{-\alpha}\mathcal{T}(n)\| \leq Ck^{\gamma+1}(n)$ for $n \geq 1$ and $\gamma \geq \alpha$ then

$$\|A_r(T)\| \leq C(1-r)^{-(\gamma-\alpha)}, \qquad 0 \leq r < 1.$$

In particular if T is a (C,α)-bounded operator, then $\sup_{0\leq r<1}\|A_r(T)\| < \infty$.

Proof. Let $\alpha \geq 0$, and $p_{\frac{1}{r}}(n) = r^{n+1}$ for $0 < r < 1$. By Remark 2.5, we have that

$$\begin{aligned} A_r(T)x &= (1-r)\sum_{n=0}^{\infty} r^n T^n(x) = \frac{1-r}{r}\sum_{n=0}^{\infty} W^{\alpha} p_{\frac{1}{r}}(n)\Delta^{-\alpha}\mathcal{T}(n)x \\ &= \frac{(1-r)^{\alpha+1}}{r}\sum_{n=0}^{\infty} p_{\frac{1}{r}}(n)\Delta^{-\alpha}\mathcal{T}(n)x = (1-r)^{\alpha+1}\sum_{n=0}^{\infty} r^n\Delta^{-\alpha}\mathcal{T}(n)x, \end{aligned}$$

where we have used Example 2.4 (i) for $0 < r < \min\{1, \frac{1}{r(T)}\}$. For $r = 0$ is obvious.

In the case that $\|\Delta^{-\alpha}\mathcal{T}(n)\| \leq Ck^{\gamma+1}(n)$ for $n \geq 1$ and $\gamma \geq \alpha$, there exists a bounded algebra homomorphism $\theta : \tau^{\alpha}(k^{\gamma+1}) \to \mathcal{B}(X)$ by Theorem 3.5. Note that $p_{\frac{1}{r}} \in \tau^{\alpha}(k^{\gamma+1})$ and $A_r(T) = \frac{1-r}{r}\theta(p_{\frac{1}{r}})$, for $0 < r < 1$. By formula (2.6), we obtain that

$$\|A_r(T)\| \leq C\frac{1-r}{r}q_{k^{\gamma+1}}(p_{\frac{1}{r}}) = C\frac{1-r}{r}\frac{r}{(1-r)^{\gamma+1-\alpha}} = \frac{C}{(1-r)^{\gamma-\alpha}}, \qquad 0 < r < 1,$$

and we conclude the proof. □

Remark 5.2. If we consider $\|T^n\| \leq Cn^{\gamma}$, with $\gamma \geq 0$, using that $n^{\gamma} \leq \Gamma(\gamma+1)k^{\gamma+1}(n)$ which follows easily from (2.3), we get that

$$\|A_r(T)\| \leq C\Gamma(\gamma+1)(1-r)^{-\gamma},$$

which improves the bound of [25, Proposition 2.1 (i) (2.3)]. Use similar arguments to improve the bound of [25, Proposition 2.1 (i) (2.4)].

Remark 5.3. An inverse result exists on Banach lattices, see [25, Corollary 3.2], which proves that for any $\alpha > -1$ and a positive bounded operator T, $\{(1-r)^{\alpha}A_r(T),\ 0 \leq r < 1\}$ is bounded if only if $\|\Delta^{-1}\mathcal{T}(n)\| \leq C(n+1)^{\alpha}$, $n \in \mathbb{N}_0$. In particular, T is Abel-mean bounded if only if is $(C,1)$-bounded. Note that there are examples of positive $(C,1)$-bounded operators in Banach lattices which are not power bounded, see remarks following [25, Corollary 3.2].

5.2. **α-Times integrated semigroups and Cesàro sums.** Now, let A be a closed linear operator on X, $\alpha > 0$ and $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ an α-times integrated semigroup generated by A, that is, $S_\alpha(0) = 0$, the map $[0,\infty) \to X$, $r \mapsto S_\alpha(r)x$ is strongly continuous and

$$S_\alpha(t)S_\alpha(s)x = \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+s}(t+s-r)^{\alpha-1}S_\alpha(r)xdr - \int_0^s (t+s-r)^{\alpha-1}S_\alpha(r)xdr\right), \qquad x \in X,$$

for $t, s > 0$; for $\alpha = 0$, $\{S_0(t)\}_{t\geq 0}$ is an usual C_0-semigroup, $S_0(0) = I$ and $S_0(t+s) = S_0(t)S_0(s)$ for $t, s > 0$. In the case that $\{S_\alpha(t)\}_{t\geq 0}$ is a non-degenerate family and $\|S_\alpha(t)\| \leq Ce^{\omega t}$ for $C > 0$,

$\omega \in \mathbb{R}$, then there exists a closed operator, $(A, D(A))$, called the generator of $\{S_\alpha(t)\}_{t\geq 0}$, such that

$$(\lambda - A)^{-1}x = \lambda^\alpha \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \qquad \Re\lambda > \omega, \qquad x \in X. \tag{5.1}$$

Moreover the following integral equality holds

$$A\int_0^t S_\alpha(s)xds = S_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x, \qquad t > 0, \quad x \in X. \tag{5.2}$$

Theorem 5.4. *Suppose that $\{S_\alpha(t)\}_{t\geq 0}$ is an α-times integrated semigroup generated by $(A, D(A))$ such that $\|S_\alpha(t)\| \leq Ce^{\omega t}$ with $0 \leq \omega < 1$. Then $1 \in \rho(A)$, $R := (1-A)^{-1}$, $\mathcal{R}(n) = R^n$ and*

$$\begin{aligned}
\Delta^{-\alpha}\mathcal{R}(n)x &= (I-A)\int_0^\infty \frac{e^{-t}t^n}{n!} S_\alpha(t)xdt, \qquad n \in \mathbb{N}_0,\\
&= \int_0^\infty \frac{e^{-t}t^{n-1}}{(n-1)!} S_\alpha(t)xdt + k^{\alpha+1}(n)x - k^{\alpha+1}(n-1)x, \qquad n \geq 1, \quad x \in X,
\end{aligned}$$

In particular if $\{S_\alpha(t)\}_{t\geq 0}$ has temperated growth, i.e. $\|S_\alpha(t)\| \leq Ct^\alpha$ for $t > 0$, then $(I-A)^{-1}$ is a (C,α)-bounded operator.

Proof. Take λ such that $\lambda \in \rho(A)$ and then

$$\frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}(\lambda^{-\alpha}(\lambda - A)^{-1}) = \sum_{j=0}^n \frac{k^\alpha(n-j)}{\lambda^{\alpha+n-j}}(\lambda - A)^{-j-1}.$$

In other hand, for λ such that $\Re\lambda > \omega$, we apply formula (5.1) to get that

$$\frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}(\lambda^{-\alpha}(\lambda - A)^{-1})x = \int_0^\infty \frac{t^n}{n!} e^{-\lambda t} S_\alpha(t)x\,dt, \qquad x \in X.$$

Finally we take $\lambda = 1$ and write $R := (1-A)^{-1}$, $\mathcal{R}(n) = R^n$ to conclude the first equality for $n \in \mathbb{N}_0$. Now for $n \geq 1$, we have that

$$\begin{aligned}
\Delta^{-\alpha}\mathcal{R}(n)x &= \int_0^\infty \frac{e^{-t}t^n}{n!} S_\alpha(t)xdt + A\int_0^\infty \frac{e^{-t}t^{n-1}}{(n-1)!}\left(1 - \frac{t}{n}\right)\int_0^t S_\alpha(s)xdsdt\\
&= \int_0^\infty \frac{e^{-t}t^{n-1}}{(n-1)!} S_\alpha(t)xdt + k^{\alpha+1}(n)x - k^{\alpha+1}(n-1)x, \qquad x \in X,
\end{aligned}$$

where we have apply the equality (5.2).

In the case that $\|S_\alpha(t)\| \leq Ct^\alpha$, we use the second equality and that the sequence $k^{\alpha+1}$ is increasing to conclude that $\sup_{n\in\mathbb{N}_0} \dfrac{\|\Delta^{-\alpha}\mathcal{R}(n)\|}{k^{\alpha+1}(n)} < \infty$ and $(I-A)^{-1}$ is a (C,α)-bounded operator. $\square$

Classical examples of generators of temperated α-times integrated semigroup are differential operators A such that their symbol $\hat{A}$ is of the form $\hat{A} = ia$ where a is a real elliptic homogeneous polynomial on $\mathbb{R}^n$ or $a \in C^\infty(\mathbb{R}^n\backslash\{0\})$ is a real homogeneous function on $\mathbb{R}^n$ such that if $a(t) = 0$ then $t = 0$, see [21, Theorem 4.2], and other different examples in [21, Section 6].

Remark 5.5. In the case of uniformly bounded C_0-semigroups, i.e. $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ such that $\sup_{t>0} \|T(t)\| < \infty$, the resolvent $(1-A)^{-1}$ is power-bounded due to

$$(1-A)^{-n}x = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-t} T(t)x dt, \qquad x \in X.$$

Note that Theorem 5.4 includes a natural extension of this fact: the resolvent $(1-A)^{-1}$ is a (C,α)-bounded operator when A generates a temperated α-times integrated semigroup.

We may also consider the homomorphism θ defined in Theorem 3.5, and in this case

$$\theta(\Delta f)x = -A\theta(f)x - (I-A)f(0)x, \qquad f \in \tau^\alpha(k^{\alpha+1}), \quad x \in D(A),$$

when A generates a temperated α-times integrated semigroup.

5.3. **Counterexamples of bounded homomorphisms.**

Example 5.6. In [9, Section 2] there is an example of a positive, Cesàro bounded but not power bounded operator T on the space ℓ^1. As the author comments in [10, Section 4. Examples], $\|T^n\|_1 \leq Kn/\ln(n)$ where K is the uniform bound of the Cesàro averages of T. In this example T is also a contraction in ℓ^∞. In [13, Section (VI)], it is proven that $\sup_{n\geq 0} \|T^n\|_p \geq (2^k)^{\frac{1}{p}}$ for any $k \geq 1$ and $1 \leq p < \infty$. We conclude that T is not power bounded in ℓ^p $(1 \leq p < \infty)$ and T is a Cesàro bounded in ℓ^p $(1 \leq p \leq \infty)$. By Corollary 3.7, there exists a bounded homomorphism $\theta : \tau^1(k^2) \to \mathcal{B}(\ell^p)$ such that $\theta(e_1) = T$ and extends to $\theta : \ell^1 \to \mathcal{B}(\ell^p)$ if and only $p = \infty$.

Example 5.7. In [27], a simple matrix construction, which unifies different approaches to the Ritt condition and ergodicity of matrix semigroups, is studied in detail. Consider the Banach space $\mathfrak{X} := X \oplus X$ with norm

$$\|x_1 \oplus x_2\|_{X\oplus X} := \sqrt{\|x_1\|^2 + \|x_2\|^2}, \qquad x_1 \oplus x_2 \in \mathfrak{X}.$$

Let the bounded linear operator $\mathfrak{T}$ on $\mathfrak{X}$ be defined by the operator matrix

$$\mathfrak{T} := \begin{pmatrix} T & T-I \\ 0 & T \end{pmatrix}$$

where $T \in \mathcal{B}(X)$. In [27, Lemma 2.1], some connected properties between T and $\mathfrak{T}$ are given. Now we consider as $X = \ell^2$ and the backward shift operator $T \in \mathcal{L}(\ell^2)$ defined by

$$T((x_n)_{n\geq 0}) := (x_{n+1})_{n\geq 0}, \qquad (x_n)_{n\geq 0} \in \ell^2.$$

By [27, Example 3.1], $\|\mathfrak{T}^n\| \geq 2n$ and $\mathfrak{T}$ is a $(C,1)$-bounded operator. We apply Corollary 3.7 to conclude that there exists an algebra homomorphisms $\theta : \tau^1(k^2) \to \mathcal{B}(\mathfrak{X})$ such $\theta(e_1) = \mathfrak{T}$ and it is not extended continuously to ℓ^1. In [27, Remark 3.2], the growth $\|\mathfrak{T}^n\| \geq 2n$ is pointed at as the fastest possible for a Cesàro bounded operator.

Example 5.8. In [25, Proposition 4.3], the following example is given. For any γ with $0 < \gamma < 1$, there exists a positive linear operator T on an L_1-space such that

$$\sup_{n\geq 0} \left\| \frac{\Delta^{-\gamma}\mathcal{T}(n)}{k^{\gamma+1}(n)} \right\| = \infty, \quad \text{but} \quad \sup_{n\geq 0} \left\| \frac{\Delta^{-\beta}\mathcal{T}(n)}{k^{\beta+1}(n)} \right\| < \infty \quad \text{for all } \beta > \gamma.$$

By Corollary 3.7, we conclude that there exists a bounded algebra homomorphism θ such that $\theta : \tau^\beta(k^{\beta+1}) \to \mathcal{B}(X)$ for all $\beta > \gamma$, $\theta(e_1) = T$, and the homomorphism θ is not extended continuously to the algebra $\tau^\gamma(k^{\gamma+1})$ with $0 < \gamma < 1$.

Example 5.9. In [25, Proposition 4.4 (i)], the following operator is constructed. Let $dimX = \infty$. For any integer $j \geq 0$, there exists a bounded linear operator T on X such that

$$\sup_{n\geq 0} \|\frac{\Delta^{-(j+1)}\mathcal{T}(n)}{k^{j+2}(n)}\| < \infty, \quad \text{but} \quad \sup_{n\geq 0} \|\frac{\Delta^{-\gamma}\mathcal{T}(n)}{k^{\gamma+1}(n)}\| = \infty \quad \text{for } 0 \leq \gamma < j+1.$$

By Corollary 3.7, we conclude that there exists a bounded algebra homomorphism θ such that $\theta : \tau^{j+1}(k^{j+2}) \to \mathcal{B}(X)$, $\theta(e_1) = T$, and the homomorphism θ is not continuously extended to the algebra $\tau^{\gamma}(k^{\gamma+1})$ with $0 \leq \gamma < j+1$.

Example 5.10. In [25, Proposition 4.4 (ii)], the following operator is constructed. Let $dimX = \infty$. There exists a bounded linear operator T on X with $r(T) = 1$, $\|T\| = 2$, and

$$\|A_r(T)\| \leq 1 - r, \quad 0 < r < 1; \quad \text{and} \quad \sup_{n\geq 0} \|\frac{\Delta^{-j}\mathcal{T}(n)}{k^{j+1}(n)}\| = \infty, \quad \text{for } j \geq 1.$$

Since $k^j(n) \leq k^{j+1}(n)$ for $n \geq 0$, we also conclude that $\|\frac{\Delta^{-j}\mathcal{T}(n)}{k^{j}(n)}\| = \infty$ for $j \geq 1$ and the converse of Theorem 5.1 does not hold for $\gamma < \alpha$.

5.4. **Application to Katznelson-Tzafriri theorem.** Let $A(\mathbb{T})$ be the regular convolution Wiener algebra formed by all continuous periodic functions $f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$, $t \in [-\pi, \pi]$, where $(\hat{f}(n))_{n\in\mathbb{Z}}$ are the Fourier coefficients of f, that is

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}\, dt, \qquad n \in \mathbb{Z},$$

with the norm $\|f\|_{A(\mathbb{T})} := \sum_{n=-\infty}^{\infty} |\hat{f}(n)|$, and $A_+(\mathbb{T})$ be the convolution closed subalgebra of $A(\mathbb{T})$ where the functions satisfies that $\hat{f}(n) = 0$ for $n < 0$. Note that both $A(\mathbb{T})$ and $\ell^1_{\mathbb{Z}}$, and $A_+(\mathbb{T})$ and ℓ^1 are isometrically isomorphic, where $\ell^1_{\mathbb{Z}}$ denotes the complex summable sequences indexed in $\mathbb{Z}$.

Katznelson and Tzafriri proved in 1986 the following well known theorem: if $T \in \mathcal{B}(X)$ is power-bounded and $f \in A_+(\mathbb{T})$ is of spectral synthesis in $A(\mathbb{T})$ with respect to $\sigma(T) \cap \mathbb{T}$, then

$$\lim_{n\to\infty} \|T^n \theta(\hat{f})\| = 0,$$

see [22, Theorem 5]. Moreover, for $T \in \mathcal{B}(X)$ a power-bounded operator, $\lim_{n\to\infty} \|T^n - T^{n+1}\| = 0$ if and only if $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$, see [22, Theorem 1].

The authors have got some similar results for (C, α)-bounded operators, which will appear in a forthcoming paper. We define $A^{\alpha}(\mathbb{T})$ a new regular Wiener algebra contained in $A(\mathbb{T})$, and $A^{\alpha}_+(\mathbb{T})$ a convolution closed subalgebra of $A^{\alpha}(\mathbb{T})$, which is isometrically isomorphic to $\tau^{\alpha}(k^{\alpha+1})$. The result prove that if $\alpha > 0$, $T \in \mathcal{B}(X)$ is a (C, α)-bounded operator and $f \in A^{\alpha}_+(\mathbb{T})$ is of spectral synthesis in $A^{\alpha}(\mathbb{T})$ with respect to $\sigma(T) \cap \mathbb{T}$, then

$$\lim_{n\to\infty} \frac{1}{k^{\alpha+1}(n)} \|\Delta^{-\alpha}\mathcal{T}(n)\theta(\hat{f})\| = 0.$$

On the continuous case, Katznelson-Tzafriri theorems have been proved for C_0-semigroups and extended later for α-times integrated semigroups, see [14] and [16] respectively.

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